Feburary 2007

Regularization for zeta functions with physical applications II

Minoru Fujimoto[1] and Kunihiko Uehara[2]

[1] *Seika Science Research Laboratory, Seika-cho, Kyoto 619-0237, Japan*
[2] *Department of Physics, Tezukayama University, Nara 631-8501, Japan*

Abstract

We have proposed a regularization technique and apply it to the Euler product of zeta functions in the part one. In this paper that is the second part of the trilogy, we give another evidence to demonstrate the Riemann hypotheses by using the approximate functional equation. Some other results on the critical line are also presented using the relations between the Euler product and the deformed summation representions in the critical strip. In part three, we will focus on physical applications using these outcomes.

PACS number(s): 02.30.-f, 02.30.Gp, 05.40.-a

[1]E-mail address: cayce@eos.ocn.ne.jp
[2]E-mail address: uehara@tezukayama-u.ac.jp

1 Introduction

In the situation that regularizations by the zeta function have been successful with some physical applications, we proposed a regularization technique[1] applicable to the Euler product representation and gave a proof for the Riemann hypotheses by using this technique in the part one. In this part, we now focus on the zeros of the Riemann zeta function and the surrounding properties of the zereos including other evidences to demonstrate the Riemann hypotheses. The Euler product representation, which played an essential role in the first part, will be interpreted in terms of the summation representation on the critical line $\Re z = \frac{1}{2}$.

The definition of the Riemann zeta function is

$$\zeta(z) = \lim_{n\to\infty} \zeta_n(z) = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k^z} = \prod_{k=1}^{\infty} \left(1 - \frac{1}{{p_k}^z}\right)^{-1} \tag{1}$$

for $\Re z > 1$, where the right hand side is the Euler product representation and p_k is the k-th prime number. Hereafter we adopt a notation $\hat{\zeta}(z)$ for $\Re z > 0$ such as

$$\hat{\zeta}(z) = \lim_{n\to\infty} \hat{\zeta}_n(z) = \lim_{n\to\infty} \frac{1}{1 - 2^{1-z}} \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^z}, \tag{2}$$

which is well regularized even in the critical strip $0 < \Re z < 1$.

Considering the approximate expansion formula for the Riemann zeta function, we propose an evience for the elegant proof of the Riemann hyposesis in section 2. And in §3 we show the surrounding properties of the zereos for the Riemann zeta function by deforming the Euler product representation to the summation form on the critical line. We study the relation between the Euler product and the summation representation in section 4, and discuss the relation between distributions of zeros and primes in connection with the Sato-Tate conjecture in §5.

2 The expansion formula and the Riemann hyposesis

The Euler-Maclaurin sum formula is given by

$$\sum_{n=M}^{N} f(n) = \int_M^N f(x)dx + \frac{1}{2}(f(M) + f(N)) + \sum_{j=1}^{k} \frac{B_{2j}}{(2j)!} \left[f^{(2j-1)}(x)\right]_M^N + R_{2k}, \tag{3}$$

where B_{2j} is the $2j$-th Bernoulli number and the remainder term:

$$\begin{aligned} R_{2k} &= \frac{1}{(2k+1)!}\int_M^N \bar{B}_{2k+1}(x) f^{(2k+1)}(x)dx, \\ \bar{B}_{2k+1}(x) &= B_{2k+1}(x-[x]). \end{aligned} \tag{4}$$

We parametrize a complex variable z by two real variable such as $z = s(\frac{1}{2} + it)$ as same as that in the first part. Using Euler-Maclaurin sum formula on the assumption of $\frac{s}{2} = \Re z > -2k$, we get the relation

$$\begin{aligned} \sum_{n=M}^{\infty} \frac{1}{n^z} &= \int_M^\infty x^{-z}dx + \frac{1}{2}M^{-z} + \int_M^\infty \bar{B}_1(x)(-z)x^{-z-1}dx \\ &= \frac{M^{1-z}}{z-1} + \frac{1}{2}M^{-z} + \sum_{j=1}^{k} \frac{B_{2j}}{(2j)!} M^{1-z-2j} \prod_{l=0}^{2j-2}(z+l) + R_{2k}, \end{aligned} \tag{5}$$

where

$$R_{2k} = \frac{-z(z+1)\cdots(z+2k)}{(2k+1)!}\int_M^\infty \bar{B}_{2k+1}(x)x^{-z-2k-1}dx. \tag{6}$$

As is well known we can go forward to the the expansion formula in the case of $M \geq 2$,

$$\zeta(z) = \sum_{n=1}^{M-1}\frac{1}{n^z} + \frac{M^{1-z}}{z-1} + \frac{1}{2}M^{-z} + \sum_{j=1}^{k}\frac{B_{2j}}{(2j)!}M^{1-z-2j}\prod_{l=0}^{2j-2}(z+l) + R_{2k}. \tag{7}$$

The remainder term R_{2k} can be estimated as follows:

$$\begin{aligned} |R_{2k}| &\leq \frac{\pi^2}{3}\left|\frac{z+2k+1}{\frac{1}{2}+2k+1}\right|\left|\frac{z(z+1)\cdots(z+2k)}{(2\pi)^{2k+2}}\right| M^{-\frac{1}{2}-2k-1} \\ &\leq C(k)\sqrt{M}\left(\frac{t}{2\pi M}\right)^{2k+2}, \end{aligned} \tag{8}$$

where we put $s = 1$ and $C(k)$ is constant only depending on k. This tells us that it is necessary for the remainder term to be $M > \frac{t}{2\pi}$ to converge.

Taking account of the Euler-Maclaurin sum formula, we can put the regularized zeta function as

$$\lim_{n\to\infty}\hat{\zeta}_n(z) = \lim_{n\to\infty}\{\zeta_n(z) - \frac{1}{1-z}n^{1-z}\} \tag{9}$$

and a zero in the critical strip is the solution to the equation

$$\hat{\zeta}(z) = \lim_{n\to\infty} \hat{\zeta}_n(z) = 0. \tag{10}$$

As $(1-\rho)$ is also the solution when ρ is the solution of Eq. (10), a solution of the equation

$$\hat{\zeta}(1-z) = 0 \tag{11}$$

is also a zero. Now we transform Eq. (9) to

$$\lim_{n\to\infty} \{(1-z)\zeta_n(z) - n^{1-z}\} = 0 \tag{12}$$

and substituting $(1-z)$ for z, we get

$$\lim_{n\to\infty} \{z\zeta_n(1-z) - n^{z}\} = 0. \tag{13}$$

Combining these equations (12) and (13), we get

$$\lim_{n\to\infty} \{z(1-z)\zeta_n(z)\zeta_n(1-z) - n\} = 0, \tag{14}$$

namely,

$$z^2 - z + \lim_{n\to\infty} \frac{n}{\zeta_n(z)\zeta_n(1-z)} = 0. \tag{15}$$

The solution ρ_n of the equation $\hat{\zeta}_n(z) = 0$ is satisfied the relation

$$\rho_n = \frac{1}{2} \pm i\sqrt{\lim_{n\to\infty} \frac{n}{\zeta_n(\rho_n)\zeta_n(1-\rho_n)}}. \tag{16}$$

On the other hand, the approximate functional equation by Hardy and Littlewood[2], which leads to the Riemann-Siegel formula, is given by

$$\zeta(z) = \sum_{n\le x} \frac{1}{n^z} + H(z)\sum_{n\le y} \frac{1}{n^{1-z}} + O(x^{-s/2}) + O(|t|^{1/2-s/2}y^{s/2-1}), \tag{17}$$

where $0 \le s/2 (= \Re z) \le 1, x \ge 1, y \ge 1, 2\pi xy = |t|$ and $H(z)$ is given by

$$H(z) = 2\Gamma(1-z)(2\pi)^{z-1}\sin\frac{\pi z}{2}. \tag{18}$$

For $s > 2$, the relation

$$\zeta(z) = H(z)\sum_{n=1}^{\infty} \frac{1}{n^{1-z}} = H(z)\zeta(1-z) \tag{19}$$

is satisfied and we write down $\hat{\zeta}(1-z)$ for $0 < \Re z < 1$ as,

$$\hat{\zeta}(1-z) = \frac{1}{1-2^z}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{1-z}}. \tag{20}$$

Then the relation

$$\hat{\zeta}(z) = H(z)\hat{\zeta}(1-z) \tag{21}$$

is satisfied for $\Re z > 0$ and substituting $(1-z)$ for z, we conclude Eq. (21) is satisfied for all z.

We set $n \leq x, n \leq y$ and $|t| = 2\pi xy \geq 2\pi n$, then

$$\hat{\zeta}(z) = \zeta_n(z) + H(z)\zeta_n(1-z) + R_n(z), \tag{22}$$

where the remainder term $R_n(z)$:

$$\begin{aligned} R_n(z) &= O(n^{-s/2}) + O(|t|^{1/2-s/2}n^{s/2-1}) \\ &= O(n^{-s/2}) + O((2\pi n)^{1/2-s/2}n^{s/2-1}) \end{aligned} \tag{23}$$

can be ignored by taking the limit of $n \to \infty$, namely

$$\lim_{n\to\infty} R_n(z) = \lim_{n\to\infty}\{O(n^{-s/2}) + O(n^{-1/2})\} = 0. \tag{24}$$

We put the relations at zeros

$$\zeta_n(\rho_n) = \frac{n^{1-\rho_n}}{1-\rho_n}, \quad \zeta_n(1-\rho_n) = \frac{n^{\rho_n}}{\rho_n} \tag{25}$$

into Eq. (22) together with $\hat{\zeta}_n(\rho_n) = 0$, we can write

$$\begin{aligned} \frac{n^{1-\rho_n}}{1-\rho_n} + H(\rho_n)\frac{n^{\rho_n}}{\rho_n} + R_n(\rho_n) &= 0, \\ \frac{n^{1-2\rho_n}}{1-\rho_n} + H(\rho_n)\frac{1}{\rho_n} + R_n(\rho_n)\frac{1}{n^{\rho_n}} &= 0. \end{aligned} \tag{26}$$

Taking the limit of $n \to \infty$, we get the relations

$$\lim_{n\to\infty}\frac{n^{1-2\rho_n}}{1-\rho_n} = \lim_{n\to\infty}\frac{-H(\rho_n)}{\rho_n} = \frac{-H(\rho)}{\rho}. \tag{27}$$

The right hand side of Eq. (27) is finite, so the numerator of the left hand side $\lim_{n\to\infty}|n^{1-2\rho_n}|$ must be finite. This means that the real part of $(1-2\rho_n)$ must converge to 0 in the limit of

$n \to \infty$ when the real part of $(1-2\rho_n)$ is positive. When $\Re(1-2\rho_n) < 0$, we rewrite the left hand side of Eq. (27) like

$$\lim_{n\to\infty} \frac{n^{1-2(1-\rho_n)}}{1-(1-\rho_n)} = \lim_{n\to\infty} \frac{n^{2\rho_n-1}}{\rho_n}, \tag{28}$$

so we get the same goal as $\lim_{n\to\infty} \Re(2\rho_n - 1) = 0$. After all, the Riemann hyposesis is satisfied including the trivial case $\Re(1-2\rho_n) = 0$,

$$\rho = \lim_{n\to\infty} \rho_n = \lim_{n\to\infty} \left(\frac{1}{2} + it_n\right) = \frac{1}{2} + i\lambda, \tag{29}$$

where λ is real but t_n is not necessarily a real number.

Now we think about the values of t_n, which converge to a positive λ in the limit of $n \to \infty$ and put it into Eq. (16)

$$\begin{aligned} \rho_n &= \frac{1}{2} + i\sqrt{\frac{n}{\zeta_n(\rho_n)\zeta_n(1-\rho_n)}} \\ &= \frac{1}{2} + i\sqrt{\frac{n}{\zeta_n(\frac{1}{2}+it_n)\zeta_n(\frac{1}{2}-it_n)}}. \end{aligned} \tag{30}$$

Thus we write the solution of the equation $\hat{\zeta}(z) = 0$

$$\begin{aligned} \rho &= \lim_{n\to\infty} \left\{ \frac{1}{2} + i\sqrt{\frac{n}{\zeta_n(\rho_n)\zeta_n(1-\rho_n)}} \right\} \\ & \quad i.e. \\ \lambda &= \lim_{n\to\infty} \sqrt{\frac{n}{\zeta_n(\frac{1}{2}+it_n)\zeta_n(\frac{1}{2}-it_n)}}. \end{aligned} \tag{31}$$

Using the n-th order relation of Eq. (21)

$$\zeta_n(\frac{1}{2} - it_n) = H(\frac{1}{2} - it_n)\zeta_n(\frac{1}{2} + it_n), \tag{32}$$

we get the relation

$$\begin{aligned} t_n &= \sqrt{\frac{n}{\zeta_n(\frac{1}{2}+it_n)H(\frac{1}{2}-it_n)\zeta_n(\frac{1}{2}+it_n)}} \\ &= \frac{1}{\zeta_n(\frac{1}{2}+it_n)}\sqrt{\frac{n}{H(\frac{1}{2}-it_n)}}. \end{aligned} \tag{33}$$

This form will be utilize to calculate zeros of the Riemann zeta function by way of the limit of $n \to \infty$.

3 The Euler product and a summation representation

We write down the Euler product representation for the standard form discribed as same as the equation (25) in the first part

$$\frac{1}{\left|\zeta_n(s(\frac{1}{2}+it))\right|^2} = f_n(s,t) = \prod_{k=1}^{n}\left(1 - \frac{2}{p_k^{s/2}}\cos(st\log p_k) + \frac{1}{{p_k}^s}\right). \tag{34}$$

Here $f_n(s,t)$ and $\log f_n(s,t)$ will diverge at the same time in $n \to \infty$, because $f_n(s,t)$ is positive. As all non-trivial zeros of the Riemann zeta function is existed on $s = 1$, we study the following relation for $s \geq 1$

$$\begin{aligned}
\log f_n(s,t) &= \sum_{k=1}^{n}\log\left(1 - \frac{2}{{p_k}^{s/2}}\cos(st\log p_k) + \frac{1}{{p_k}^s}\right) \\
&= \sum_{k=1}^{n}\log\left\{1 - \left(\frac{2}{{p_k}^{s/2}}\cos(st\log p_k) - \frac{1}{{p_k}^s}\right)\right\} \\
&= -\sum_{k=1}^{n}\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{2}{{p_k}^{s/2}}\cos(st\log p_k) - \frac{1}{{p_k}^s}\right)^m \\
&= -\sum_{k=1}^{n}\left(\frac{2}{{p_k}^{s/2}}\cos(st\log p_k) - \frac{1}{{p_k}^s}\right) - \frac{1}{2}\sum_{k=1}^{n}\left(\frac{2}{{p_k}^{s/2}}\cos(st\log p_k) - \frac{1}{{p_k}^s}\right)^2 \\
&\quad -\frac{1}{3}\sum_{k=1}^{n}\left(\frac{2}{{p_k}^{s/2}}\cos(st\log p_k) - \frac{1}{{p_k}^s}\right)^3 + (\text{finite terms in } n \to \infty).
\end{aligned} \tag{35}$$

We must regularize Eq. (35) in order to apply it even in the case $s = 1$. We try to reularize Eq. (35) by way of dividing an appropriate factor $\sum_{k=1}^{n}\frac{1}{p_k}$, which leaves a leading divergence divergent and makes a non-leading divergence convergent. In fact, we divide Eq. (35) by

$$\prod_{k=1}^{n}\left(1 + \frac{1}{p_k}\right), \tag{36}$$

which we also adopted in the equation (26) of the first part. Thus we study the divergence

in the form of

$$\frac{\displaystyle\prod_{k=1}^{n}\left(1-\frac{2}{p_k^{s/2}}\cos(st\log p_k)+\frac{1}{{p_k}^s}\right)}{\displaystyle\prod_{k=1}^{n}\left(1+\frac{1}{p_k}\right)}, \tag{37}$$

which corresponds in the summation form as

$$\frac{\displaystyle -2\sum_{k=1}^{n}\frac{\cos(t\log p_k)}{\sqrt{p_k}}\left(\frac{\cos(t\log p_k)}{\sqrt{p_k}}+1\right)+\sum_{k=1}^{n}\frac{1}{p_k}}{\displaystyle\sum_{k=1}^{n}\frac{1}{p_k}}, \tag{38}$$

where we set $s=1$. The leading divergent term of Eq. (38) in $n\to\infty$ is

$$\sum_{k=1}^{n}\frac{\cos(t\log p_k)}{\sqrt{p_k}}. \tag{39}$$

The form of divisor $\displaystyle\prod_{k=1}^{n}\left(1+\frac{1}{p_k}\right)$ means that using the Mertens' theorem

$$\prod_{k=1}^{m}\left(1-\frac{1}{p_k}\right)=\frac{e^{-\gamma}}{\log p_m}\left(1+O\left(\frac{1}{\sqrt{p_m}}\right)\right) \tag{40}$$

and the Euler's $\zeta(2)$

$$\prod_{k=1}^{\infty}\left(1-\frac{1}{{p_k}^2}\right)=\frac{6}{\pi^2}, \tag{41}$$

we get

$$
\begin{aligned}
C\prod_{k=1}^{m}\left(1+\frac{1}{p_k}\right) &= \prod_{k=1}^{m}\frac{\left(1-\frac{1}{{p_k}^2}\right)}{\left(1-\frac{1}{p_k}\right)}\prod_{k=m+1}^{\infty}\left(1-\frac{1}{{p_k}^2}\right) \\
&= \frac{6e^{\gamma}}{\pi^2}\log p_m\left(1+O\left(\frac{1}{\sqrt{p_m}}\right)\right), \qquad (42) \\
\log\prod_{k=1}^{m}\left(1+\frac{1}{p_k}\right) &= \sum_{k=1}^{m}\log\left(1+\frac{1}{p_k}\right) \\
&= \sum_{k=1}^{m}\frac{1}{p_k}-\frac{1}{2}\left(\sum_{k=1}^{m}\frac{1}{p_k}\right)^2+\cdots \\
&\simeq \log\log p_m, \qquad (43)
\end{aligned}
$$

where

$$\frac{6e^{\gamma}}{\pi^2}\leq C\leq e^{\gamma}. \qquad (44)$$

The Euler product representation for $n\to\infty$ is only valid for $s\geq 2$, and we restrict our interest for $t>0$. The zeros of the Riemann zeta function make Eq. (34) divergent, which means that products are multiplied maximally in the right hand side. Each term in Eq. (34) is maximized when $\cos(st\log p_k)=-1$, namely, $st=\dfrac{(2m-1)\pi}{\log p_k}$ (m =natural number). We give graphs for the superposition of cosine functions, which indicate the solution of $\cos(t\log p_k)=-1$ as local maximum values,

$$y_{n.\alpha}(t)=\sum_{k=1}^{n}-\frac{\cos(t\log p_k)}{{p_k}^{\alpha}}. \qquad (45)$$

Judging from the graph of $\alpha=1$ (Figure 2), the denominator p_k seems to be well-matched to cancell the notches come from the superposition of cosines. Thus zeros of the Euler product representation in Eq. (34) preserve the value even in the form of the summation in Eq. (45). The terms to regularize the divergence will be formed essential to the order on the critical line in the next section.

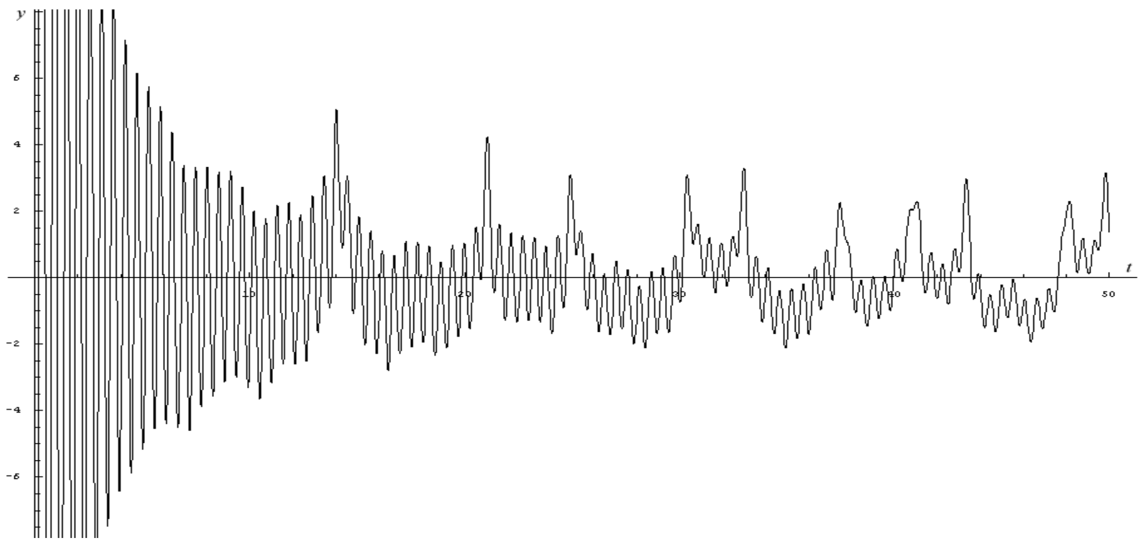

Figure 1: The graph of $y_{n,\alpha}$ for $n = 10^4$ and $\alpha = \frac{1}{2}$.

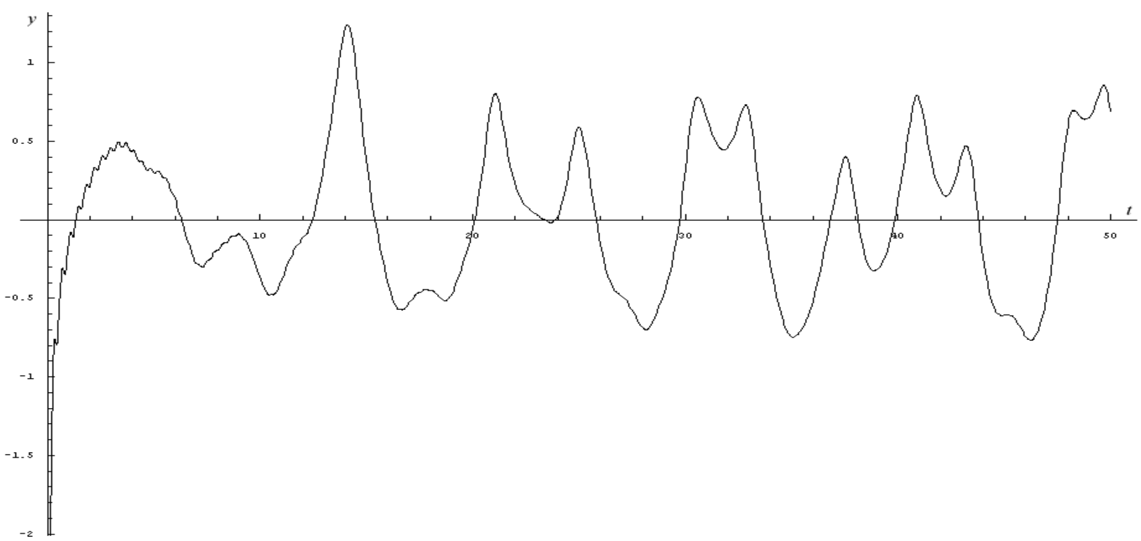

Figure 2: The graph of $y_{n,\alpha}$ for $n = 10^6$ and $\alpha = 1$.

4 The order of the Euler product for the Riemann zeros

Here we introduce two notations as

$$\zeta_n^{\Sigma}(z) \equiv \sum_{k=1}^{n} \frac{1}{k^z} \tag{46}$$

$$\zeta_m^{\Pi}(z) \equiv \prod_{k=1}^{m} \frac{1}{1-{p_k}^{-z}} \tag{47}$$

to clarify the relation between n of the summation form and m of the product form.

$$\begin{aligned}
\zeta_m^{\Pi}(z) &\equiv \prod_{k=1}^{m} \frac{1}{1-{p_k}^{-z}} \\
&= \left(1+\frac{1}{2^m}+\frac{1}{2^{2m}}+\frac{1}{2^{3m}}+\cdots\right)\left(1+\frac{1}{3^m}+\frac{1}{3^{2m}}+\frac{1}{3^{3m}}+\cdots\right) \\
&\quad \times\left(1+\frac{1}{5^m}+\frac{1}{5^{2m}}+\frac{1}{5^{3m}}+\cdots\right)\cdots\left(1+\frac{1}{{p_m}^z}+\frac{1}{{p_m}^{2z}}+\frac{1}{{p_m}^{3z}}+\cdots\right) \\
&= 1+\frac{1}{2^m}+\frac{1}{3^m}+\cdots+\frac{1}{{p_m}^z}+\frac{1}{(p_m+1)^z}+\cdots,
\end{aligned} \tag{48}$$

where we point out that the terms after (p_m+1) are composite numbers. We can easily get the relation

$$\zeta_n^{\Sigma}(z) < \zeta_n^{\Pi}(z). \tag{49}$$

As $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k^z}$ converges for the value $\Re z > 1$, we can write down by

$$\sum_{k=1}^{m} \frac{1}{k^z} = O(1) \tag{50}$$

for k which is a subset of a natural number. We can write

$$\sum_{k=2}^{\infty} \frac{1}{k^z} = \zeta(z) - 1, \tag{51}$$

so it reads

$$\sum_{l=2}^{\infty}\sum_{k=2}^{\infty} \frac{1}{k^l} = \sum_{l=2}^{\infty}\{\zeta(l)-1\} = 1. \tag{52}$$

As for a convergence of Eq. (52), k can be replaced by $\sqrt{k}$ or $\sqrt{p_k}$, so long as we add up with k from $k = 3$.

Thus we only deal with a series of $\prod(1 - \frac{1}{\sqrt{p_n}})$ with fisrt power or less,

$$\begin{aligned}
\zeta_m^{\Pi}(1/2) &= \prod_{k=1}^{m} \frac{1}{1-(\sqrt{p_k})^{-1}} \\
&= \prod_{k=1}^{m} \left(1 + \frac{1}{\sqrt{p_k}} + \frac{1}{p_k} + \frac{1}{p_k\sqrt{p_k}} + \frac{1}{{p_k}^2} + \frac{1}{{p_k}^2\sqrt{p_k}} + \cdots\right) \\
&= \prod_{k=1}^{m} \left(1 + \frac{1}{\sqrt{p_k}} + \frac{1}{p_k} + \frac{1}{p_k\sqrt{p_k}} + \frac{1}{{p_k}^2}\frac{1}{1-(\sqrt{p_k})^{-1}}\right) \\
&= 1 + \sum_{k=1}^{m} \frac{1}{\sqrt{p_k}} + \sum_{k=1}^{m} \frac{1}{p_k} + \sum_{k=1}^{m} \frac{1}{p_k\sqrt{p_k}} + \sum_{k=1}^{m} \frac{1}{{p_k}^2}\frac{1}{1-(\sqrt{p_k})^{-1}} \\
&\quad + \sum_{k,l=1,k\neq l} \left(\frac{1}{p_k\sqrt{p_l}} + \frac{1}{p_l\sqrt{p_k}}\right) + \cdots \\
&= 1 + \sum_{k=1}^{m} \frac{1}{\sqrt{p_k}} + \sum_{k=1}^{m} \frac{1}{p_k} + O(1) \\
&= 1 + \sum_{k=1}^{m} \frac{1}{\sqrt{k\log k}} + \log p_m + O(1). \qquad (53)
\end{aligned}$$

We focus on the second term

$$\begin{aligned}
\sum_{k=1}^{m} \frac{1}{\sqrt{p_k}} &= \frac{1}{\sqrt{2}} + \sum_{k=2}^{m} \frac{1}{\sqrt{p_k}} = \frac{1}{\sqrt{2}} + \sum_{k=2}^{m} \frac{1}{k\log k} + O(1) \\
&= \int_3^{m\log m} \frac{1}{\sqrt{r\log r}} dr + O(1). \qquad (54)
\end{aligned}$$

We estimate the infinite integral of the first term and get the series

$$\begin{aligned}
\int \frac{1}{\sqrt{r\log r}} dr &= 2\sqrt{r}\left(\frac{1}{\sqrt{\log r}} + \frac{1}{\sqrt{(\log r)^3}} + \frac{1\cdot 3}{\sqrt{(\log r)^5}} + \cdots + \frac{1\cdot 3\cdots(2n-3)}{\sqrt{(\log r)^{2n-1}}}\right) \\
&\quad + \int \frac{1\cdot 3\cdots(2n-1)}{\sqrt{r(\log r)^{2n+1}}} dr \qquad (55)
\end{aligned}$$

As the r value of last term corresponds $m\log m$, we put $r = m\log m$ into $2\sqrt{\frac{r}{\log r}} + O(1)$

$$
\begin{aligned}
2\sqrt{\frac{m\log m}{\log(m\log m)}} + o(\sqrt{m}) &= 2\sqrt{\frac{m\log m}{\log m + \log\log m}} + o(\sqrt{m}) \\
&= 2\sqrt{\frac{m}{1+\frac{\log\log m}{\log m}}} + o(\sqrt{m}) \\
&= 2\sqrt{m} + o(\sqrt{m}) \qquad (56)
\end{aligned}
$$

From Eq. (46), we estimate

$$
\zeta_n^{\Sigma}(1/2) = \sum_{k=1}^{n} \frac{1}{\sqrt{k}} = \int_1^n \frac{dk}{\sqrt{k}} + O(1) = 2\sqrt{m} + O(1) \qquad (57)
$$

Thus we conclude

$$
\begin{aligned}
\{\zeta_n^{\Pi}(1/2)\}^2 &= 4n + o(\sqrt{n}) \\
\lim_{n\to\infty} \frac{\zeta_n^{\Pi}(1/2)}{\zeta_n^{\Sigma}(1/2)} &= 1. \qquad (58)
\end{aligned}
$$

These equations reads

$$
\lim_{n\to\infty} \frac{\zeta_n^{\Pi}(1/2+it_n)\zeta_n^{\Pi}(1/2-it_n)}{\zeta_n^{\Sigma}(1/2+it_n)\zeta_n^{\Sigma}(1/2-it_n)} = 1, \qquad (59)
$$

which means that the order is n. This fact confirms the value of $\alpha = 1$ in Eq. (45) for the smooth. In this way we can substitute $\zeta_n^{\Pi}(z)$ for $\zeta_n^{\Sigma}(z)$ in Eq. (31)

$$
\lambda = \lim_{n\to\infty} \sqrt{\frac{n}{\zeta_n^{\Pi}(\frac{1}{2}+it_n)\zeta_n^{\Pi}(\frac{1}{2}-it_n)}}. \qquad (60)
$$

5 Discussions and remarks

Here we can show that the Riemann hypothsis holds for the L-function by using the approximate functional equation for the Dirichlet's L-function as well as that by using the regularization for the Euler product as stated in part one. We listed the condition which leads the verification for the Riemannn hypotheses as

1. the existence of the Euler product representation,

2. the prime number theorem $\pi(x) \simeq \dfrac{x}{\log x}$ is satisfied,

3. the approximate functional equation of the Dirichlet's L-function is satisfied.

Misuses of the Euler-Maclaurin expansion for the zeta function, which is actually an asymptotic expansion, have prevented the Riemann hypothesis from being demonstrated.

According to the conclusion of the first part, the Riemann hypothsis for the Ramanujan's zeta function or another zeta function is realized because each function has the Euler product representation. The Ramanujan's conjecture for the Euler product corresponds the cosine term of the standard form for the Riemann zeta function, so it will hold because $|\cos\theta|$ is less than one due to the independence of $\log p_k$.

About the zeta functions, which have no non-trivial zero besides zeros of the Riemann hypotheses, we parametrize them to the standard form. In such a case that the Sato-Tate theorem for the elliptical zeta function shown by Richard Taylor has a same structure for uniform distributions[3] for the zeros λ_i of the Riemann zeta function and $\log p_k$, the logarithm of the primes p_k, the Sato-Tate conjectures[4] for all zeta function that have the standard form of the Euler product will be realized. Moreover the conjecture will hold for the direction of zeros λ_i as well as primes p_k.

The Sato-Tate conjecture, the 2θ of which corresponds $\lambda_j \log p_k$ here, claims that the response of k-direction increase($k = 1, 2, \cdots, \infty$) of p_k yields the distribution of

$$\frac{2}{\pi} \int_{\alpha}^{\beta} \sin^2 \theta d\theta, \tag{61}$$

where $0 \leq \alpha \leq \theta \leq \beta \leq \pi$. On the other hand, once we can put $2\theta = \lambda_j \log p_k$, we claim that the response of j-direction increase($j = 1, 2, \cdots, \infty$) of λ_j yields the same distribution above as Eq. (61). The relation above is satisfied when zeros λ_j or prime numbers p_k of $\lambda_j \log p_k (= 2\theta)$ is taken so that $\cos 2\theta$ is close to -1.

We normalize the product of λ_j and $\log p_k$ introducing new notations μ_j and ν_k as

$$\mu_j = \lambda_j, \quad \nu_k = \frac{\log p_k}{2\pi}, \tag{62}$$

and the k-direction($k = 1, 2, \cdots, \infty$) average of $\mu_j \nu_k - \frac{1}{2} - [\mu_j \nu_k]$ will be 0 from the Sato-Tate conjecture, where [ ] is the Gauss symbol. By the law of large number, we can write down

$$\sum_{k=1}^{m} \left(\mu_j \nu_k - \frac{1}{2} \right) - \sum_{k=1}^{m} [\mu_j \nu_k] = O\left(\frac{1}{m} \right), \tag{63}$$

so we get

$$\mu_j = \frac{1}{\sum_{k=1}^{m} \nu_k} \left\{ \sum_{k=1}^{m} [\mu_j \nu_k] + \frac{m}{2} + O\left(\frac{1}{m}\right) \right\}. \tag{64}$$

We can estimate the denominator as[5]

$$\log p_k = \log(k \log k + O(\log k)) = \log k + O(\log\log k), \tag{65}$$

$$\nu_k = \frac{1}{2\pi} \log p_k = \frac{1}{2\pi}(\log k + O(\log\log k)), \tag{66}$$

$$\begin{aligned} \sum_{k=1}^{m} \nu_k &= \sum_{k=1}^{m} \frac{1}{2\pi} \log p_k \\ &= \frac{1}{2\pi} \int_1^m (\log x + O(\log\log x))dx \\ &= \frac{1}{2\pi} m(\log m - 1) + O((\log\log m)\log m). \end{aligned} \tag{67}$$

After all, we write a following approximate relation for any μ_j,

$$\begin{aligned} \mu_j &= \frac{\sum_{k=1}^{m} [\mu_j \nu_k] + \frac{m}{2} + O\left(\frac{1}{m}\right)}{\frac{1}{2\pi} m(\log m - 1) + O((\log\log m)\log m)} \\ &= \frac{2\pi \sum_{k=1}^{m} [\mu_j \nu_k]}{m(\log m - 1)} + \frac{\pi}{\log m - 1} + O\left(\frac{1}{m^2 \log m}\right), \end{aligned} \tag{68}$$

using $\lambda_j = \mu_j, \nu_k = \dfrac{\log p_k}{2\pi}$, we write the relation for any λ_j

$$\lambda_j = \frac{2\pi \sum_{k=1}^{m} \left[\frac{\lambda_j \log p_k}{2\pi}\right]}{m(\log m - 1)} + \frac{\pi}{\log m - 1} + O\left(\frac{1}{m^2 \log m}\right). \tag{69}$$

In the similar way, we take the j-direction average of $\mu_j \nu_k - \frac{1}{2} - [\mu_j \nu_k]$, we can write down

$$\sum_{j=1}^{n} \left(\mu_j \nu_k - \frac{1}{2}\right) - \sum_{j=1}^{n} [\mu_j \nu_k] = O\left(\frac{1}{n}\right) \tag{70}$$

so we get

$$\nu_k = \frac{1}{\sum_{j=1}^{n} \mu_j} \left\{ \frac{n}{2} + \sum_{j=1}^{n} [\mu_j \nu_k] + O\left(\frac{1}{n}\right) \right\} \tag{71}$$

We also estimate the demoninator as

$$\begin{aligned} \lambda_j &= \frac{2\pi j}{\log j + \log 2\pi} + O\left(\frac{j}{\log^2 j}\right) && (72) \\ \sum_{j=1}^{n} \lambda_j &= \lambda_1 + 2\pi \sum_{j=2}^{n} \frac{j}{\log j} + \sum_{j=2}^{n} O\left(\frac{j}{\log^2 j}\right) \\ &= 2\pi \int_2^n \frac{x}{\log x} dx + \int_2^n O\left(\frac{x}{\log^2 x}\right) dx \\ &= 2\pi \left[\frac{x^2}{2\log x}\right]_2^n + \pi \int_2^n \frac{x}{\log^2 x} dx + \left[O\left(\frac{j}{\log^2 j}\right)\right]_2^n \\ &= \frac{\pi n^2}{\log n} + O\left(\frac{n^2}{\log^2 n}\right) = \sum_{j=1}^{n} \mu_j && (73) \end{aligned}$$

so we write a following approximate relation for any ν_k,

$$\begin{aligned} \nu_k &= \frac{\sum_{j=1}^{n} [\mu_j \nu_k] + \frac{n}{2} + O\left(\frac{1}{n}\right)}{\frac{\pi n^2}{\log n} + O\left(\frac{n}{\log^2 n}\right)} \\ &= \frac{1}{2\pi} \left(\frac{2\log n \sum_{j=1}^{n} [\mu_j \nu_k]}{n^2} + \frac{\log n}{n} \right) + O\left(\frac{\log n}{n^3}\right). && (74) \end{aligned}$$

Finally we write down the relation for any p_k

$$\log p_k = 2\pi\nu_k = \frac{2\log n \sum_{j=1}^{n}\left[\frac{\lambda_j \log p_k}{2\pi}\right]}{n^2} + \frac{\log n}{n} + O\left(\frac{\log n}{n^3}\right) \tag{75}$$

$$p_k = \exp\left\{\frac{2\log n \sum_{j=1}^{n}\left[\frac{\lambda_j \log p_k}{2\pi}\right]}{n^2} + \frac{\log n}{n}\right\} \cdot \exp\left\{O\left(\frac{\log n}{n^3}\right)\right\}$$

$$= n^{\left(\frac{2\sum_{j=1}^{n}\left[\lambda_j \log p_k/(2\pi)\right]}{n^2}+\frac{1}{n}\right)} \cdot n^{O\left(\frac{1}{n^3}\right)} \tag{76}$$

Equations (69) and (76) are a set of equations which gives prime numbers and zeros of the Riemann zeta function.

The equation

$$\sum_{j=1}^{M}\sum_{k=1}^{N} \mu_j\nu_k - [\mu_j\nu_k] - \frac{1}{2} = 0 \tag{77}$$

for sufficient large numbers M and N, which we call the function equation for the zeta functions, will be the equations for primes and zeros. In the similar way, we can think about another function equation

$$\sum_{j=1}^{M}\sum_{k=1}^{N} \mu_j\nu_k - [\mu_j\nu_k] = 0, \tag{78}$$

which is lead to from the zeta function

$$f(z) = \prod_{k=1}^{\infty}(1 + {p_k}^{-s})^{-1} = \frac{\zeta(2z)}{\zeta(z)}. \tag{79}$$

The summation representation of this kind zeta function is

$$f(z) = \sum_{n=1}^{\infty}\frac{\lambda(n)}{n^{-z}}, \tag{80}$$

where $\lambda(n)$ is the the Dirichlet lambda function defined by

$$\lambda(n) = \sum_{k=0}^{\infty}\frac{1}{(2k+1)^n} = (1 - 2^{-n})\zeta(n) \tag{81}$$

which is just the form substituted $\lambda(n)$ for $\tau(n)$ of the Ramanujan's zeta function. The function $f(z)$ has zeros at $2z = \frac{1}{2} \pm i\lambda_j$, namely, $z = \frac{1}{4} \pm i\frac{\lambda_j}{2}$ and has poles at $z = \frac{1}{2} \pm i\lambda_j$.

The zeta function above in Eq. (79) has a functional equation, so the Riemann hypothesis associated with it is realized. The trivial zeros, which are negative even numbers, for the Riemann zeta function is realized for $\mu = -2m$ and $nu = 1(\text{mod } \frac{1}{2m})$

When we think about the function

$$g_n(z) = \prod_{k=1}^{n} \frac{\zeta(2^k z)}{\zeta(2^{k-1} z)}, \tag{82}$$

which may have the similar nature as for zeros and poles to s-type zeta function like

$$\zeta_{SL(2,Z)}(z) = \prod_{p \in \mathrm{Prim}(SL(2,Z))} \{1 - N(p)^{-z}\}^{-1}, \tag{83}$$

which is known as the Selberg zeta function. In the limit of $n \to \infty$ for $g_n(z)$, $g_\infty(z)$ will converge to $\zeta(z)^{-1}$.

References

[1] M. Fujimoto and K. Uehara, Regularization for zeta functions with physical applications I, arXiv:math-ph/0609013 (2006).

[2] J. B. Corey, D. W. Farmer, J. P. Keating, M. O. Rubinstein and N. C. Snaith, Integral moments of L-functions, Proceedings of the London Mathematical Society 91, Cambridge University Press (2005), 33-104.

[3] P. D. T. A. Elliot, The Riemann zeta function and coin tossing, J. Reine Angew. Math. 254 (1972), 100-109.

[4] B. Mazur, Controlling our errors, Nature 443 (2006), 38-40.

[5] H. M. Srivastava and J.Choi, Series Associated with the Zeta and Related Function, Kluwer (2001).